Notes on Logical Expression for Brandom's Seminar

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1 Logical Expressivism: A Precisification

Definition 1.1. Brandom understands expressivism in terms of what he calls an "LX relation", where a vocabulary B is "LX" of a vocabulary A if it is elaborated from and explicative of A. The first criterion (elaboration) has it that if one is able to successfully deploy vocabulary A then one already has the skills necessary to use B. That is, that B may be (algorithmetically) elaborated from A. The second criterion (explication) has it that B says something about (makes perspicuous in the object language) what one was doing by using A (minimally that B may encode the implications and incompatibilities of A). Logical vocabulary is said to be "universally LX" meaning that logical vocabulary stands in this relation to *all vocabularies*.

Definition 1.2 (Expression 1)**.** Fix a logic $\mathbb{L}$, i.e. a function from $\succ_0$ to $\succ$. We say that $\mathbb{L}$ is **expressive** or that $\succ$ **expresses** a base consequence relation $\succ_0$ iff:

$$\begin{aligned} &(\forall \Gamma, \Theta \subseteq \mathcal{L})(\exists \Gamma_1, \Theta_1, \dots, \Gamma_n, \Theta_n \subseteq \mathcal{L}_0) \\ (\forall {\succ_0} \subseteq \mathcal{P}(\mathcal{L}_0)^2)&(\forall {\succ} \subseteq \mathcal{P}(\mathcal{L})^2(\mathbb{L} : {\succ_0} \mapsto {\succ})) \\ &((\Gamma \succ \Theta) \Leftrightarrow (\Gamma_1 \succ_0 \Theta_1 \bigwedge \Gamma_2 \succ_0 \Theta_2 \bigwedge \cdots \bigwedge \Gamma_n \succ_0 \Theta_n)). \end{aligned}$$

We also say $\Gamma \succ \Theta$ **expresses** $\Gamma_i \succ_0 \Theta_i$ $(1 \leq i \leq n)$ (its expressientia).

Definition 1.3. Let $\mathfrak{S}\mathfrak{f}$ be a structural feature. Officially, a structural feature is a property of elements of consequence relations. So we can think of $\mathfrak{S}\mathfrak{f}$ as a family of partitions, one for each member of the family of consequence relations (i.e. a partition for each subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$).[1] Let $\mathfrak{S}\mathfrak{f}(\Gamma \succ \Theta)$ be shorthand for $\Gamma \succ \Theta$ obeys (is an instance of) $\mathfrak{S}\mathfrak{f}$. Next, let $\Gamma \succ \Theta$ be arbitrary with $\Gamma_i \succ_0 \Theta_i$ $(1 \leq i \leq n)$ its expressientia (in accordance with Definition 1.2). We say that a logic $\mathbb{L}$ **preserves** a structural feature $\mathfrak{S}\mathfrak{f}$ iff:

$$\mathfrak{S}\mathfrak{f}(\Gamma \succ \Theta) \Leftrightarrow \left(\mathfrak{S}\mathfrak{f}(\Gamma_1 \succ_0 \Theta_1) \bigwedge \cdots \bigwedge \mathfrak{S}\mathfrak{f}(\Gamma_n \succ_0 \Theta_n)\right).$$

Definition 1.4 (Expression 2)**.** Let $\mathfrak{S}\mathfrak{f}$ be a structural feature. Suppose some logical operation '*' may be used to mark a sequent in some way (with the constraint that $\Gamma^* \succ \Theta^*$ only if $\Gamma \succ \Theta$). Then we say that '*' (or $\mathbb{L}$) expresses $\mathfrak{S}\mathfrak{f}$ iff there exists a '*' in $\mathbb{L}$ such that:

$$\Gamma^* \succ \Theta^* \Leftrightarrow \mathfrak{S}\mathfrak{f}(\Gamma \succ \Theta).$$

[1] Unofficially, a structural feature usually has something like an intension associated with it. It may be that the monotonic and transitive consequences of a consequence relation are the same. Nevertheless, the characterization of each is different, because the extensions may differ on a different consequence relation.

Sf-Expression combines three ideas. (i) That a logic be capable of expressing an underlying base consequence relation, (ii) that it be capable of preserving structural features of that base consequence relation, and finally (iii) that it be able to mark in the object language those very same features that it preserves.

I want to show two things:

1. I introduce a logic that is expressively complete in the sense of Expression 1 (Def 1.2)

2. The same logic is also expressively complete in the sense of Expression 2 (Def 1.4). All structural features that are preserved may be expressed.

1.1 A Note On Structural Features (Re: Second Kind of Expression)

Typically consequence relations (the thing that we want to express (features of)) are assumed to be fully structural, i.e.:

- Monotonic
- Transitive
- Contractive
- Reflexive

Non-Monotonicity

$$A_1 = \text{“This piece of pasta is long and cylindrical”}$$
$$A_2 = \text{“This piece of pasta is spaghetti”}$$

Clearly,

$$A_1 \vdash A_2$$

$$B_1 = \text{“This piece of pasta is smooth, tubular, and straight-cut”}$$
$$B_2 = \text{“This piece of pasta is cannelloni”}$$

Clearly,

$$B_1 \vdash B_2$$

Next, suppose C = "This piece of pasta is bucatini". Clearly,

$$A_1, B_1 \nvdash A_2$$
$$A_1, B_1 \nvdash B_2$$
$$A_1, B_1 \vdash C$$

Non-Transitivity

C_1 ="This is a piece of pasta is bucatini"
C_2 ="This piece of pasta is long (*pasta lunga*)"
C_3 ="This piece of pasta could be made by rolling and cutting (lasagna type)"

Clearly,

$$C_1 \vdash C_2$$
$$C_2 \vdash C_3$$
$$C_1 \nvdash C_3$$

Non-Contractiveness & Non-Reflexiveness

Won't be motivated today...

Further Considerations

- Central commitment of inferentialists (non-representationalist) that content have two dimensions (role in premises and conclusions; "implicit content" and "explicit content"). All of the structural features run these notions together, i.e. collapse role in premises and conclusions.
- To be clearer: to specify what follows from a sentence A would be sufficient to characterize its content given all of those structural features.

Conceptually: closed consequence relation; fully structural consequence relations must be rejected

2 Introducing NM-MS

Definition 2.1 (Base Consequence Relation (BCR))**.** A base consequence relation is a relation between finite multi-sets of atomic sentences, e.g. $\vdash_0 \subseteq \mathcal{P}(\mathcal{L}_0)^2$ (where $\mathcal{L}_0$ is the set of all atomic sentences of the language).

Start with "material" base consequence relation, i.e. a base consequence relation with no further constraints on it.

To be clear, none of the following constraints on a base consequence relation:

Definition 2.2 (Constraints on a base)**.** A base consequence relation may satisfy certain constraints.

Reflexive: A BCR is **reflexive** iff, for all $p \in \mathcal{L}_0$: $p \vdash_0 p$.

Containment: A BCR satisfies **containment** iff, for all $p \in \mathcal{L}_0$ all $\langle \Delta, \Lambda \rangle \in \mathcal{P}(\mathcal{L}_0)^2$ we have: $\Delta, p \vdash_0 p, \Lambda$.

Monotonicity: A BCR is **monotonic** iff for all $\langle \Gamma, \Theta \rangle$ and $\langle \Delta, \Lambda \rangle$ in $\mathcal{P}(\mathcal{L}_0)^2$ we have: if $\Gamma \vdash_0 \Theta$ then $\Delta, \Gamma \vdash_0 \Theta, \Lambda$.

Contractive: A BCR is **contractive** iff $A, A, \Gamma \vdash_0 \Theta$ only if $A, \Gamma \vdash_0 \Theta$ and $\Gamma \vdash_0 \Theta, A, A$ only if $\Gamma \vdash_0 \Theta, A$ (for arbitrary Γ, Θ, A).

Transitive: A BCR is **transitive** iff $A, \Gamma \vdash_0 \Theta$ and $\Gamma \vdash_0 \Theta, A$ only if $\Gamma \vdash_0 \Theta$ (for arbitrary Γ, Θ, A.[2]

Next, I define a consequence relation, $\vdash$ as any sequent derivable from the sequent calculus below, where the leaves are generated from a single Axiom (given a particular BCR $\vdash_0$).

Axiom: If $\Gamma \vdash_0 \Theta$ then $\Gamma \vdash \Theta$.

$$\frac{\Gamma \vdash \Theta, A \qquad B, \Gamma \vdash \Theta}{A \rightarrow B, \Gamma \vdash \Theta}\ \text{L}{\rightarrow} \qquad \frac{A, \Gamma \vdash \Theta, B}{\Gamma \vdash A \rightarrow B, \Theta}\ \text{R}{\rightarrow}$$

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A \& B \vdash \Theta}\ \text{L\&} \qquad \frac{\Gamma \vdash \Theta, A \qquad \Gamma \vdash \Theta, B}{\Gamma \vdash \Theta, A \& B}\ \text{R\&}$$

[2]We can define a similar version for mixed-cut. I do not in this document.

$$\frac{A, \Gamma \vdash \Theta \qquad B, \Gamma \vdash \Theta}{A \lor B, \Gamma \vdash \Theta}\,\text{L}\lor \qquad \frac{\Gamma \vdash \Theta, A, B}{\Gamma \vdash \Theta, A \lor B}\,\text{R}\lor$$

$$\frac{\Gamma \vdash \Theta, A}{\neg A, \Gamma \vdash \Theta}\,\text{L}\neg \qquad \frac{A, \Gamma \vdash \Theta}{\Gamma \vdash \Theta, \neg A}\,\text{R}\neg$$

2.1 A Special Case for Containment

- As inferentialist we believe that the meaning of a sentence is determined by its role in implication
- As expressivists we believe that the role of logical vocabulary is to express such roles and the relationships between them
- Thus, if two sentences have the same meaning (i.e. participate in the same implications identically, i.e. can be substituted for one another:

$$A, \Gamma \vdash \Delta \Leftrightarrow B, \Gamma \vdash \Delta$$
$$\Gamma \vdash \Delta, A \Leftrightarrow \Gamma \vdash \Delta, B$$

then this ought to be expressible:

$$\forall \Gamma, \Delta \subseteq \mathcal{L}(\Gamma, A \vdash B\Delta)$$
$$\forall \Gamma, \Delta \subseteq \mathcal{L}(\Gamma, B \vdash A\Delta)$$

Proposition 2.3. Suppose A and B have the same meaning (i.e. can be substituted for one another). Then they imply each other in arbitrary contexts:

$$\forall \Gamma, \Delta \subseteq \mathcal{L}(\Gamma, A \vdash B, \Delta)$$
$$\forall \Gamma, \Delta \subseteq \mathcal{L}(\Gamma, B \vdash A, \Delta)$$

iff CO holds:

$$\forall \Gamma, \Delta \subseteq \mathcal{L} \forall C \in \mathcal{L}(\Gamma, C \vdash C, \Delta)$$

Proof. ($\Rightarrow$) We get CO as a special case where $A = B$.

($\Leftarrow$) Suppose CO holds, then $\Gamma, A \vdash A, \Delta$ for arbitrary Γ, Δ. Since A and B can be intersubstituted clearly:

$$(\Gamma, A \vdash B\Delta)$$
$$(\Gamma, B \vdash A\Delta)$$

∎

2.2 Some Features of NM-MS

Theorem 2.4. *If* $\Gamma \vdash \Theta$ *may be arbitrarily weakened with atoms, then it may be arbitrarily weakened with logically complex sentences:*

$$\forall \Delta_0, \Lambda_0 \subseteq \mathcal{L}_0(\Delta_0, \Gamma \vdash \Theta, \Lambda_0) \Leftrightarrow \forall \Delta, \Lambda \subseteq \mathcal{L}(\Delta, \Gamma \vdash \Theta, \Lambda).$$

Proof. ($\Leftarrow$) is immediate. ($\Rightarrow$) is proven by induction on the complexity of $\Delta \cup \Lambda$ where complexity is understood in terms of the complexity of the most complex sentences in $\Delta \cup \Lambda$. ■

Theorem 2.5. *If* $\Gamma \vdash \Theta$ *allows contraction of atomic sentences, then it allows contraction of logically complex sentences.*

Proof. One direction is trivial, the other direction is provided by induction on the complexity of the contracted sentence. ■

Since it is well known that the rules featured above are equivalent to both the additive and multiplicative rules of linear logic given contraction and monotonicity, we can actually locate the condition needed for our logic to be supra-classical.

Definition 2.6. We say that $\vdash_0$ obeys Containment (CO) if

$$\forall \Delta, \Lambda \subseteq \mathcal{L}_0(\Delta, p \vdash_0 p, \Lambda)$$

(i.e. if we have $\forall q \in \mathcal{L}_0(q \vdash_0 q)$ and all such sequents may be arbitrarily weakened; the fragment carved out by this stipulation will also obviously obey contraction). In short: let us define $\vdash_0^{CO}$ such that $\vdash_0^{CO}$ obeys reflexivity $\forall q \in \mathcal{L}_0(q \vdash_0 q)$, weakening and contraction. And further stipulate that no proper subset of $\vdash_0^{CO}$ obeys all of these conditions. A base consequence relation $\vdash_0$ is said to obey CO iff it includes $\vdash_0^{CO}$, i.e. $\vdash_0^{CO} \subseteq \vdash_0$.

Theorem 2.7. *If* $\vdash_0$ *obeys CO, then* $\vdash$ *is supra-classical.*

Proof. Result is well known, but can be easily established by showing an equivalence with Gentzen's LK in the fragment of $\vdash$ generated by $\vdash_0^{CO}$. ■

Finally, the next theorem is of particular import to the sections following this one.

Theorem 2.8. *All rules of the sequent calculus are reversible. That is, if* $\Gamma \vdash \Theta$ *would be the result of the application of a rule to* $\Gamma^* \vdash \Theta^*$ *(and possibly* $\Gamma^{**} \vdash \Theta^{**}$*) then*

$$\Gamma \vdash \Theta \Leftrightarrow \Gamma^* \vdash \Theta^* (and \quad \Gamma^{**} \vdash \Theta^{**}).$$

Proof. Proof is straightforward by induction on proof height. ∎

From this my first gloss on logical expression follows immediately. In the next section I prove that the more precise sense (in Definition 1.2) also holds.

Corollary 2.9. *The sequent calculus is conservative. That is*

$$\Gamma \vdash_0 \Theta \Leftrightarrow \Gamma \vdash \Theta.$$

3 Expression 1 & Expressive Completeness

Next I show how consequence relations may be represented in NM-MS. First two central results concerning conjunctive and disjunctive normal forms.

Proposition 3.1. Let $CNF(A)$ be the conjunctive normal form representation of A. It follows that

$$\Gamma \vdash \Theta, A \Leftrightarrow \Gamma \vdash \Theta, CNF(A).$$

Proof. Proof proceeds constructively. From theorem 2.8, we may deconstruct A until we have a number of sequents of the form: $\Gamma \vdash \Theta, \Lambda_1$; $\Gamma \vdash \Theta, \Lambda_2; \ldots \Gamma \vdash \Theta, \Lambda_n$ where $\Lambda_i (1 \leq i \leq n)$ contains only literals. We next construct $CNF(A)$ via repeated application of R∨ and R&:

$$\Gamma \vdash \Theta, (\bigvee \Lambda_1) \& (\bigvee \Lambda_2) \& \cdots \& (\bigvee \Lambda_n),$$

i.e. $\Gamma \vdash \Theta, CNF(A)$. ∎

Proposition 3.2. Let $DNF(A)$ be the disjunctive normal form representation of A. It follows that

$$A, \Gamma \vdash \Theta \Leftrightarrow DNF(A), \Gamma \vdash \Theta.$$

Proof. Proof is identical to the previous proposition except the sets are on the left and we construct $DNF(A)$ via L& and L∨. ∎

Theorem 3.3 (Representation Theorem 1)**.** *Let CR be a consequence relation, i.e. $CR \subseteq \mathcal{P}(\mathcal{L})^2$. Then we may specify what must be included in $\vdash_0$ such that $CR \subseteq \vdash$.*

Proof. Proof proceeds constructively. For each $\Gamma \vdash \Theta$ in CR let us find an equivalent $CNF(A) \vdash CNF(B)$. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \vdash (\bigvee \Theta_1) \& \cdots \& (\bigvee \Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \vdash_0 \Theta_j$. Thus we stipulate of the base that $\Gamma_i \vdash_0 \Theta_j$ for $1 \leq i \leq a$ and $1 \leq j \leq b$. If we do this for each implication in CR then we are guaranteed that $CR \subseteq \vdash$. ∎

Theorem 3.4 (Representation Theorem 2)**.** *Let CR be a consequence relation. If CR is closed under some modest syntactic constraints,*[3] *then we may specify* $\vdash_0$ *such that* $CR = \vdash$.

Proof. Proof is identical to the first Representation Theorem except that the syntactic constraints on CR have it that $\vdash = CR$. ■

These results give us a way of saying exactly how to reconstruct arbitrary consequence relations using my machinery and given some modest constraints how to reconstruct them *exactly*. It is this ability to reconstruct consequence relations *exactly* that will prove most important. For what it shows is that we are able to find exactly which pre-logical implications an arbitrary implication involving logical vocabulary *expresses*. That is, what I have shown is a method for finding exactly which implications in $\vdash_0$ are expressed by each implication in $\vdash$. We are thus in a position to prove the following straight away.

Theorem 3.5 (Expressive Completeness)**.** *NM-MS is expressive. That is, we have*

$$\Gamma \vdash \Theta \Leftrightarrow (\Gamma_1 \vdash_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \vdash_0 \Theta_n).$$

for some $\Gamma_1, \Theta_1, \ldots, \Gamma_n, \Theta_n$ *and arbitrary* $\vdash_0$.

Proof. Suppose $\Gamma \vdash \Theta$ and let it be equivalent to $DNF(A) \vdash CNF(B)$ for some A and B. This has the form:

$$(\&\Gamma_1) \vee \cdots \vee (\&\Gamma_a) \vdash (\bigvee\Theta_1)\& \cdots \&(\bigvee\Theta_b).$$

This holds just in case (for $1 \leq i \leq a$ and $1 \leq j \leq b$) $\Gamma_i \vdash_0 \Theta_j$. Next, let us enumerate $\langle i, j\rangle$ as $1, \ldots, n$. Then we have that:

$$\Gamma \vdash \Theta \Leftrightarrow (\Gamma_1 \vdash_0 \Theta_1 \bigwedge \cdots \bigwedge \Gamma_n \vdash_0 \Theta_n).$$

■

[3] In a more formal account I treat representation as of *theories* (see the appendix). Here I characterize it in terms of consequence relations, where we are able to precisely represent a consequence relation just in case it is closed under the rules of NM-MS. In the case where we wish to treat theories instead, then a theory T must meet the following constraints: &-composition and -decomposition ($A, B \in T$ iff $A\&B \in T$), Distribution (of $\vee$ over &) $(A \vee (B\&C)) \in T$ iff $(A \vee B)\&(A \vee C) \in T$, Conditional Equivalence ($A \to B = \sigma$ is a sub-formula of $\tau \in T$ iff $\neg A \vee B = \sigma'$ is a subformula of $\tau \in T$), both De-Morgan's Equivalences (likewise defined over sub-formulae) and involution (also defined over subformulae).

4 Expression 2 & Expressive Completeness 2

First, let us enrich our sequent calculus by introducing a second turnstile $\vdash^{\mathfrak{S}}$. Now let $\vdash_0^{\mathfrak{S}}$ pick out some subset of $\vdash_0$. Later I will discuss principles for determining which subset, but for now I leave the details vague. We may introduce the following rules to our sequent calculus:[4]

Axiom 2: If $\Gamma \vdash_0^{\mathfrak{S}} \Theta$ then $\Gamma \vdash^{\mathfrak{S}} \Theta$.

$$\frac{A, \Gamma \vdash^{\mathfrak{S}} \Theta}{\boxed{\mathfrak{S}}A, \Gamma \vdash^{[\mathfrak{S}]} \Theta}\ \text{L}\boxed{\mathfrak{S}} \qquad\qquad \frac{\Gamma \vdash^{\mathfrak{S}} \Theta, A}{\Gamma \vdash^{[\mathfrak{S}]} \Theta, \boxed{\mathfrak{S}}A}\ \text{R}\boxed{\mathfrak{S}}$$

Lemma 4.1. *L$\boxed{\mathfrak{S}}$ and R$\boxed{\mathfrak{S}}$ are reversible rules.*

We thus have the following result.

Theorem 4.2 (Expressive Completeness 2)**.** *Let $\mathfrak{Sf}$ be a structural rule. Suppose that $\mathfrak{Sf}$ is* preserved *(in the sense of Definition 1.3) and suppose further that $\vdash^{\mathfrak{S}}$ marks that structural feature exactly. We thus have: $\mathfrak{Sf}(\Gamma \vdash \Theta)$ iff $\Gamma \vdash^{\mathfrak{S}} \Theta$. It follows that $\boxed{\mathfrak{S}}$* expresses *(in the sense of Definition 1.4) $\mathfrak{Sf}$. Thus:*

$$\boxed{\mathfrak{S}}A, \Gamma \vdash \Theta \Leftrightarrow \mathfrak{Sf}(A, \Gamma \vdash \Theta)$$
$$\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(A, \Gamma \vdash \Theta, A)$$

Proof. I prove only the latter biconditional since the proof of the former is identical. By supposition $\mathfrak{Sf}(\Gamma \vdash \Theta, A)$ iff $\Gamma \vdash^{\mathfrak{S}} \Theta, A$. Since it follows that our R$\boxed{\mathfrak{S}}$ rule is reversible, we have that $\Gamma \vdash^{\mathfrak{S}} \Theta, A$ iff $\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A$. Thus

$$\Gamma \vdash \Theta, \boxed{\mathfrak{S}}A \Leftrightarrow \mathfrak{Sf}(\Gamma \vdash \Theta, A).$$

■

The result of the above proof is a general method for introducing logical vocabulary that is *expressive* of structural features. If the rules for the logical vocabulary's introduction are reversible and the structural feature in question is *preserved* by $\mathbb{L}$, then the logical vocabulary will *express* that structural feature. I next rehearse two specific cases of this: an operator that marks monotonicity and an operator that marks classical validity.

[4] Note that the rest of our sequent calculus is altered such that our other rules preserve $\vdash^{\mathfrak{S}}$. E.g. R& requires that both top sequents have either $\vdash$ or $\vdash^{\mathfrak{S}}$ (I do not allow mixed cases).

4.1 Expressing Structural Features

The rules for monotonicity have the following form:

Axiom 2: If $\forall\Delta,\Lambda \subseteq \mathcal{L}_0(\Delta,\Gamma \vdash_0 \Theta,\Lambda)$ then $\Gamma \vdash^M \Theta$.

$$\frac{A,\Gamma \vdash^M \Theta}{\boxed{M}A,\Gamma \vdash^{[M]} \Theta}\,\text{L}\boxed{M} \qquad \frac{\Gamma \vdash^M \Theta, A}{\Gamma \vdash^{[M]} \Theta, \boxed{M}A}\,\text{R}\boxed{M}$$

I have already shown in Theorem 2.4 that weakening is preserved by the rules of NM-MS. It therefore follows that:

Corollary 4.3. $\boxed{M}$ *expresses weakening/monotonicity. That is,*

$$\boxed{M}A,\Gamma \vdash \Theta \Leftrightarrow \forall\Delta,\Lambda(\Delta, A,\Gamma \vdash \Theta,\Lambda)$$
$$\Gamma \vdash \Theta,\boxed{M}A \Leftrightarrow \forall\Delta,\Lambda(\Delta,\Gamma \vdash \Theta, A,\Lambda)$$

This means that we may expand NM-MS (our $\mathbb{L}$) in order to mark *in the object language* which implications are persistent under arbitrary weakenings. Next, I show a similar result for contraction. That is, I show a way of marking sequents that are where contraction holds. We introduce the following axiom and rules as before:

Axiom 2: If $\Gamma \vdash_0 \Delta$ and for arbitrary Δ,Λ we have $\Delta \vdash_0 \Lambda$ if this would be the result of some number of applications of contraction to $\Gamma \vdash_0 \Delta$, then $\Gamma \vdash^C \Theta$.

$$\frac{A,\Gamma \vdash^C \Theta}{\boxed{C}A,\Gamma \vdash^{[C]} \Theta}\,\text{L}\boxed{C} \qquad \frac{\Gamma \vdash^C \Theta, A}{\Gamma \vdash^{[C]} \Theta, \boxed{C}A}\,\text{R}\boxed{C}$$

I have already shown in Theorem 2.5 that contraction is preserved by the rules of NM-MS. It therefore follows that:

Corollary 4.4. $\boxed{C}$ *expresses contraction.*

$$\boxed{C}A,\Gamma \vdash \Theta \Leftrightarrow (A \in \Gamma \Rightarrow \Gamma \vdash \Theta)$$
$$\Gamma \vdash \Theta,\boxed{C}A \Leftrightarrow (A \in \Theta \Rightarrow \Gamma \vdash \Theta)\,.$$

Alternatively and equivalently, if $\boxed{C}A,\Gamma \vdash \Theta$ *and* $\Gamma' \vdash \Theta'$ *is the result of some number of applications of contraction to* $A,\Gamma \vdash \Theta$ *then* $\boxed{C}A,\Gamma \vdash \Theta$ *iff* $\Gamma' \vdash \Theta'$ *(and likewise for the right-hand side).*

Next, I demonstrate the same for "classicality", i.e. develop an operator that marks implications that are valid classically.

Axiom 2: If $\Gamma, p \vdash_0 p,\Theta$ then $\Gamma, p \vdash^K p,\Theta$ (where Γ,Θ may be possibly empty).

$$\frac{A, \Gamma \vdash^{K} \Theta}{\boxed{K}A, \Gamma \vdash^{[K]} \Theta}\ \text{L}\boxed{K} \qquad \frac{\Gamma \vdash^{K} \Theta, A}{\Gamma \vdash^{[K]} \Theta, \boxed{K}A}\ \text{R}\boxed{K}$$

Again, I have already shown in Theorem 2.7 that classicality is a feature NM-MS preserves. Thus any sequent which is derived from atomic sequents which are part of the CO (cf. Definition 2.6) fragment of $\vdash_0$ (regardless of whether $\vdash_0$ actually obeys CO) will be classically valid.

Corollary 4.5. *Let $\vdash_{LK}$ be the consequence relation instantiated by Gentzen's LK minus the rules for quantifiers (and with $\wedge$ substituted with &, etc.). Then $\boxed{K}$ expresses classical validity, that is:*

$$\boxed{K}A, \Gamma \vdash \Theta \Leftrightarrow A, \Gamma \vdash_{LK} \Theta$$
$$\Gamma \vdash \Theta, \boxed{K}A \Leftrightarrow \Gamma \vdash_{LK} \Theta, A$$

There are of course many further possibilities for such '$\boxed{\mathfrak{S}}$' operators. We may also introduce vocabulary for expressing inference that obey, transitivity + weakening, more restricted weakening principles, and perhaps more.

4.2 Some Defective Cases

So far I have introduced a more precise criterion for understanding logical expressivism and in particular for understanding how *structural* features of inference might be expressed. I then introduced a system that was not only *expressive* in this sense, but also successfully preserved and expressed several important structural features. In order to appreciate exactly what I am up to, however, it will be useful to look at some cases where each of these criteria fail.

Example 4.6. The multiplicative rules of linear logic are *not expressive.* I show that this is the case for the multiplicative conjunction $\otimes$:

$$\frac{\Gamma, A, B \vdash \Theta}{\Gamma, A \otimes B \vdash \Theta}\ \text{L}\otimes \qquad \frac{\Gamma \vdash \Theta, A \qquad \Delta \vdash \Lambda, B}{\Gamma, \Delta \vdash \Theta, \Lambda, A \otimes B}\ \text{R}\otimes$$

It is sufficient to show a case where the logic does not express particular implications in $\vdash_0$. Notice that there are potentially two ways to derive $p \otimes q \vdash p \otimes q$ where $p, q \in \mathcal{L}_0$:

$$\dfrac{\dfrac{p \vdash p \qquad q \vdash q}{p, q \vdash p \otimes q}\ \text{R}\otimes}{p \otimes q \vdash p \otimes q}\ \text{L}\otimes \qquad \dfrac{\dfrac{p, q \vdash q}{p \otimes q \vdash q}\ \text{L}\otimes \qquad \vdash p}{p \otimes q \vdash p \otimes q}\ \text{R}\otimes$$

Since the atomic sequents used to start each proof tree are different (in fact they are entirely different), it's possible that $\vdash_0$ includes one and $\vdash_0'$ includes the other and thus the presence of $p \otimes q \vdash p \otimes q$ does not guarantee the presence of either. In this sense, logics which include '$\otimes$' are not expressive in the relevant sense.

It is also possible to find counter-examples to Sf-Preservation and Sf-Expression. Even using the rules of NM-MS such counter-examples will arise:

Example 4.7. Suppose we want to introduce an operator '$\boxed{R}$' to mark instances of reflexivity, i.e. $\phi \vdash \phi$. Then the rules for introducing such an operator should probably have the form:

Axiom 2: If $p \vdash_0 p$ then $p \vdash^R p$.

$$\frac{A, \Gamma \vdash^R \Theta}{\boxed{R}A, \Gamma \vdash^{[R]} \Theta}\ \text{L}\boxed{R} \qquad\qquad \frac{\Gamma \vdash^R \Theta, A}{\Gamma \vdash^{[R]} \Theta, \boxed{R}A}\ \text{R}\boxed{R}$$

Unfortunately, it is easy to show that NM-MS fails to preserve reflexivity and thus fails to express it. For example $A\&B \vdash A\&B$ is clearly an instance of reflexivity and thus we should want $A\&B \vdash \boxed{R}(A\&B)$. But clearly $A\&B \vdash A\&B$ must be derived from $A, B \vdash A$ and $A, B \vdash B$, neither of which are instances of reflexivity.[5]

There will therefore be logics which in general fail to be expressive and even among those that are expressive there will be structural features that fail to be preserved and thus expressed. Deciding how expressive one wants one's logic to be and which structural features ought to be preserved are therefore *not* independent questions.

[5]Though they are both found in the region of the consequence relation which allows reflexivity *together with weakening*, hence why we are able to have an operator to mark classicality.

It is also worth remarking that the above might also fail for independent reasons. For example, if we are able to derive $A\&B \vdash A\&B$, then we could also derive $A\&B \vdash B\&A$, but is the latter here an instance of the structural feature of reflexivity? It is not obvious that we should think so. In general, even when a sequent calculus preserves reflexivity, it needn't generate *only* reflexive sequents from the reflexive fragment of its axioms.